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Abstract. In this paper, several characterizations of fuzzy strongly irresolvable spaces are studied. The conditions under which fuzzy strongly irresolvable spaces become fuzzy first category spaces, fuzzy Baire spaces and fuzzy σ-Baire spaces are also investigated.

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Keywords: Fuzzy dense set, Fuzzy nowhere dense set, Fuzzy G_δ-set, Fuzzy F_σ-set,
Fuzzy first category space, Fuzzy Baire space, Fuzzy almost resolvable space.

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1. Introduction

The concept of fuzzy set was introduced by Zadeh [19] in 1965 as a new approach for modeling uncertainties. The potential of fuzzy notion was realized by the researchers and has successfully been applied in all branches of mathematics. In mathematics, topology provided the most natural frame work for the concepts of fuzzy sets to flourish. The concept of fuzzy topological space was introduced by Chang [4] in 1968. The paper of Chang paved the way for the subsequent tremendous growth of the numerous fuzzy topological concepts. Several mathematicians have tried all the pivotal concepts of general topology for extension to the fuzzy settings and thus a modern theory of fuzzy topology has been developed. Today, fuzzy topology has been firmly established as one of the basic disciplines of fuzzy mathematics. The systematic study of resolvability began with the works of Hewitt [7] and Katetov [8]. Resolvable and irresolvable spaces were studied extensively by Hewitt. In 1963, Ceder [5] introduced maximally resolvable spaces. El'Kin [6] introduced open hereditarily irresolvable spaces in classical topology. The concept of almost resolvable spaces was introduced and studied by Bolstein [3] as a generalization of resolvable spaces. The concept of strongly irresolvable spaces was introduced by Hewitt and it was also studied extensively by Rose et al. [9]. The concept of strongly irresolvable spaces in fuzzy setting was introduced by Thangaraj and Seenivasan [16]. The aim of this paper is to study several characterizations of fuzzy strongly irresolvable spaces. The conditions under which fuzzy strongly irresolvable spaces become fuzzy first category spaces, fuzzy Baire spaces and fuzzy σ-Baire spaces are also investigated.

2. Preliminaries

In order to the exposition self-contained, we introduce some basic notions and results used in the sequel. In this work by (X, T) or simply by X we will denote a fuzzy topological space due to Chang (1968). Let X be a nonempty set and I the unit interval [0,1]. A fuzzy set λ in X is a mapping from X into I

Definition 2.1 ([4])**.** A fuzzy topology is a family 'T'of fuzzy sets in a set X which satisfies the following conditions:

(i) $0_X, 1_X \in T$,
(ii) if $\lambda, \beta \in T$, then $\lambda \wedge \beta \in T$,
(iii) if $\lambda_i \in T$, for each $i \in I$, then $\vee_{i\in I}\lambda_i \in T$.

T is called a fuzzy topology for X and the pair (X, T) is a fuzzy topological space or fts, in short. The members of T is called T-open fuzzy sets or simply fuzzy open sets. A fuzzy set is fuzzy closed iff its complement is fuzzy open.

Definition 2.2 ([4])**.** Let λ and μ be fuzzy sets in X. Then for all $x \in X$,

(i) $\lambda = \mu \Leftrightarrow \lambda(x) = \mu(x)$,
(ii) $\lambda \leq \mu \Leftrightarrow \lambda(x) \leq \mu(x)$,
(iii) $\psi = \lambda \vee \mu \Leftrightarrow \psi(x) = \max\{\lambda(x), \mu(x)\}$,
(iv) $\delta = \lambda \wedge \mu \Leftrightarrow \delta(x) = \min\{\lambda(x), \mu(x)\}$,
(v) $\eta = \lambda^c \Leftrightarrow \eta(x) = 1 - \lambda(x)$.

For a family $\{\lambda_i \mid i \in I\}$ of fuzzy sets in X, the union $\psi = \vee_i\lambda_i$ and intersection $\delta = \wedge_i\lambda_i$ are defined by $\psi(x) = \sup_i\{\lambda_i(x) \mid x \in X\}$ and $\delta(x) = \inf_i\{\lambda_i(x) \mid x \in X\}$.

Definition 2.3 ([4])**.** Let (X, T) be any fuzzy topological space and λ be any fuzzy set in (X, T). The closure and interior of a fuzzy set λ in a fuzzy topological space (X, T) are respectively denoted as $cl(\lambda)$ and $int(\lambda)$ and defined as

(i) $cl(\lambda) = \wedge\{\mu \mid \lambda \leq \mu, 1 - \mu \in T\}$ and
(ii) $int(\lambda) = \vee\{\mu \mid \mu \leq \lambda, \mu \in T\}$.

Lemma 2.4 ([1])**.** *For a fuzzy set λ of a fuzzy space X,*

(1) $1 - cl(\lambda) = int(1 - \lambda)$ *and*
(2) $1 - int(\lambda) = cl(1 - \lambda)$.

Definition 2.5 ([2])**.** A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy F_σ-set in (X, T), if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where $1 - \lambda_i \in T$, for $i \in I$.

Definition 2.6 ([2])**.** A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy G_δ-set in (X, T), if $\lambda = \wedge_{i=1}^{\infty}(\lambda_i)$, where $\lambda_i \in T$, for $i \in I$.

Definition 2.7 ([13])**.** A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy dense set, if there exists no fuzzy closed set μ in (X, T) such that $\lambda < \mu < 1$. That is, $cl(\lambda) = 1$, in (X, T).

Definition 2.8 ([13])**.** Let (X, T) be a fuzzy topological space. A fuzzy set λ in (X, T) is called a fuzzy nowhere dense set, if there exists no non-zero fuzzy open set μ in (X, T) such that $\mu < cl(\lambda)$. That is, $int\ cl(\lambda) = 0$, in (X, T).

Definition 2.9 ([13])**.** Let (X, T) be a fuzzy topological space. A fuzzy set λ in (X, T) is called a fuzzy first category set, if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy nowhere dense sets in (X, T). Any other fuzzy set in (X, T) is said to be of fuzzy second category.

Definition 2.10 ([13])**.** Let λ be a fuzzy first category set in a fuzzy topological space (X, T). Then $1 - \lambda$ is called a fuzzy residual set in (X, T).

Lemma 2.11 ([1])**.** *For a family $\mathcal{A} = \{\lambda_a\}$ of fuzzy sets of a fuzzy space X, Then, $\vee\ cl(\lambda_\alpha) \leq cl(\vee\lambda_\alpha)$. In case $\mathcal{A}$ is a finite set, $\vee\ cl(\lambda_\alpha) = cl(\vee\lambda_\alpha)$. Also $\vee\ int\ (\lambda_\alpha) \leq int(\vee\lambda_\alpha)$.*

Definition 2.12 ([15])**.** Let (X, T) be a fuzzy topological space. A fuzzy set λ in (X, T) is called a fuzzy σ-nowhere dense set, if λ is a fuzzy F_σ-set in (X, T) such that $int(\lambda) = 0$.

Definition 2.13 ([15])**.** Let (X, T) be a fuzzy topological space. A fuzzy set λ in (X, T) is called a σ-fuzzy first category set, if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy σ-nowhere dense sets in (X, T). Any other fuzzy set in (X, T) is said to be of fuzzy σ-second category.

Definition 2.14 ([10])**.** A fuzzy set λ in a fuzzy topological space X is called

(i) fuzzy pre-open, if $\lambda \leq intcl(\lambda)$ and fuzzy pre-closed if $clint(\lambda) \leq \lambda$,
(ii) fuzzy semi-open, if $\lambda \leq clint(\lambda)$ and fuzzy semi-closed if $intcl(\lambda) \leq \lambda$.

Definition 2.15 ([11])**.** A fuzzy topological space (X, T) is said to be a fuzzy Baire space, if $int\,(\vee_{i=1}^{\infty}(\lambda_i)) = 0$, where (λ_i)'s are fuzzy nowhere dense sets in (X, T).

Definition 2.16 ([15])**.** A fuzzy topological space (X, T) is said to be a fuzzy σ-Baire space, if $int\,(\vee_{i=1}^{\infty}(\lambda_i)) = 0$, where (λ_i)'s are fuzzy σ-nowhere dense sets in (X, T).

Definition 2.17 ([13])**.** A fuzzy topological space (X, T) is called a fuzzy first category space, if the fuzzy set 1_X is a fuzzy first category set in (X, T). That is, $1_X = \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy nowhere dense sets in (X, T). Otherwise, (X, T) will be called a fuzzy second category space.

Definition 2.18 ([15])**.** A fuzzy topological space (X, T) is called a fuzzy σ-first category space, if the fuzzy set 1_X is a fuzzy σ-first category set in (X, T). That is, $1_X = \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy σ-nowhere dense sets in (X, T). Otherwise, (X, T) will be called a fuzzy σ-second category space.

Definition 2.19 ([12])**.** A fuzzy topological space (X, T) is called a fuzzy nodec space, if each non-zero fuzzy nowhere dense set λ is fuzzy closed in (X, T). That is, if λ is a fuzzy nowhere dense set in (X, T), then $1 - \lambda \in T$.

Definition 2.20 ([14])**.** A fuzzy topological space (X, T) is called a fuzzy resolvable space, if there exists a fuzzy dense set λ in (X, T) such that $1 - \lambda$ is also a fuzzy dense set in (X, T). Otherwise (X, T) is called a fuzzy irresolvable space.

Definition 2.21 ([17]). A fuzzy topological space (X, T) is called a fuzzy almost resolvable space, if $\vee_{i=1}^{\infty}(\lambda_i) = 1$, where the fuzzy sets (λ_i)'s in (X, T) are such that $int(\lambda_i) = 0$. Otherwise (X, T) is called a fuzzy almost irresolvable space.

Definition 2.22 ([18]). A fuzzy topological space (X, T) is called a fuzzy σ-resolvable space, if $\vee_{i=1}^{\infty}(\lambda_i) = 1$, where (λ_i)'s are fuzzy dense sets in (X, T) such that $\lambda_i \leq (1 - \lambda_j)$, for $i \neq j$. A fuzzy topological space, which is not a fuzzy σ-resolvable space, is called a fuzzy σ-irresolvable space.

Theorem 2.23 ([11]). *If λ is a fuzzy dense and fuzzy open set in a fuzzy topological space (X, T), then $1 - \lambda$ is a fuzzy nowhere dense set in (X, T).*

Theorem 2.24 ([11]). *Let (X, T) be a fuzzy topological space. Then the following are equivalent:*

(1) *(X, T) is a fuzzy Baire space,*
(2) *$int(\lambda) = 0$, for every fuzzy first category set λ in (X, T),*
(3) *$cl(\mu) = 1$, for every fuzzy residual set μ in (X, T).*

Theorem 2.25 ([17]). *If a fuzzy topological space (X, T) is a first category and fuzzy nodec space, then (X, T) is a fuzzy almost resolvable space.*

Theorem 2.26 ([16]). *The following assertions are equivalent for a fuzzy topological space (X, T):*

(1) *(X, T) is a fuzzy strongly irresolvable space,*
(2) *if $int(\lambda) = 0$, for any non-zero fuzzy set λ in (X, T), then $intcl(\lambda) = 0$.*

3. Fuzzy strongly irresolvable spaces

Definition 3.1. A fuzzy topological space (X, T) is called a fuzzy strongly irresolvable space, if for every fuzzy dense set λ in (X, T), $clint(\lambda) = 1$ in (X, T). That is, $cl(\lambda) = 1$ implies that $clint(\lambda) = 1$, in (X, T).

Proposition 3.2. *If $clint(\lambda) \neq 1$, for every fuzzy set λ in a fuzzy strongly irresolvable space (X, T), then $cl(\lambda) \neq 1$ in (X, T).*

Proof. Let λ be a fuzzy set in (X, T) such that $clint(\lambda) \neq 1$. It is to be proved that $cl(\lambda) \neq 1$ in (X, T). Suppose that $cl(\lambda) = 1$ in (X, T). Since (X, T) is a fuzzy strongly irresolvable space, $cl(\lambda) = 1$ implies that $clint(\lambda) = 1$. This is a contradiction. Then it must be that $cl(\lambda) \neq 1$ in (X, T). □

Proposition 3.3. *If λ is a fuzzy set in a fuzzy strongly irresolvable space (X, T) such that $intcl(\lambda) \neq 0$, then $int(\lambda) \neq 0$ in (X, T).*

Proof. Let λ be a fuzzy set in (X, T) such that $intcl(\lambda) \neq 0$ in (X, T). It has to be proved that $int(\lambda) \neq 0$ in (X, T). Suppose that $int(\lambda) = 0$ in (X, T). Then $cl(1 - \lambda) = 1 - int(\lambda) = 1 - 0 = 1$. Since (X, T) is a fuzzy strongly irresolvable space, $cl(1 - \lambda) = 1$ implies that $clint(1 - \lambda) = 1$. Thus $1 - intcl(\lambda) = 1$ in (X, T). That is, $intcl(\lambda) = 0$. This is a contradiction. So it must be that $int(\lambda) \neq 0$ in (X, T). □

Proposition 3.4. *If λ is a fuzzy dense and fuzzy G_δ-set in a fuzzy strongly irresolvable space (X,T), then $(1-\lambda)$ is a fuzzy nowhere dense set and a fuzzy σ-nowhere dense set in (X,T).*

Proof. Let λ be a fuzzy dense and fuzzy G_δ-set in (X,T). Since (X,T) is a fuzzy strongly irresolvable space, for the fuzzy dense set λ, $clint(\lambda)=1$ in (X,T). Then $1-clint(\lambda)=0$. Thus $intcl(1-\lambda)=0$ in (X,T). So $(1-\lambda)$ is a fuzzy nowhere dense set in (X,T). But $int(1-\lambda)\leq intcl(1-\lambda)$. Hence $int(1-\lambda)\leq 0$. That is, $int(1-\lambda)=0$ in (X,T). Since λ is a fuzzy G_δ-set, $(1-\lambda)$ is a fuzzy F_σ-set in (X,T). Then $(1-\lambda)$ is a fuzzy F_σ-set in (X,T) such that $int(1-\lambda)=0$ and so $(1-\lambda)$ is a fuzzy σ-nowhere dense set in (X,T). Therefore $(1-\lambda)$ is a fuzzy nowhere dense set and a fuzzy σ-nowhere dense set in (X,T). □

Proposition 3.5. *If $\lambda_i\leq(1-\lambda_j)$, $(i\neq j)$, where λ_i is a fuzzy dense set in a fuzzy strongly irresolvable space (X,T), then λ_j is a fuzzy nowhere dense set in (X,T).*

Proof. Let λ_i and λ_j, $(i\neq j)$ be fuzzy sets in (X,T) such that $\lambda_i\leq(1-\lambda_j)$ and $cl(\lambda_i)=1$ in (X,T). Since (X,T) is a fuzzy strongly irresolvable space, for the fuzzy dense set λ_i, $clint(\lambda_i)=1$ in (X,T). Now $\lambda_i\leq(1-\lambda_j)$ implies that $clint(\lambda_i)\leq clint(1-\lambda_j)$. Then $1\leq clint(1-\lambda_j)$. That is, $clint(1-\lambda_j)=1$. Thus $1-intcl(\lambda_j)=1$. So $intcl(\lambda_j)=0$ implies that λ_j is a fuzzy nowhere dense set in (X,T). □

Proposition 3.6. *If each fuzzy dense set λ is a fuzzy first category set in a fuzzy strongly irresolvable space (X,T), then $intcl\left(\wedge_{i=1}^{\infty}(1-\lambda_i)\right)=0$, where (λ_i)'s are fuzzy nowhere dense sets in (X,T).*

Proof. Let λ be a fuzzy dense set in (X,T). Since (X,T) is a fuzzy strongly irresolvable space, $cl(\lambda)=1$ implies that $clint(\lambda)=1$ in (X,T). Suppose that λ is a fuzzy first category set in (X,T). Then $\lambda=\vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy nowhere dense sets in (X,T). Thus $clint(\lambda)=1$ implies that $clint\left(\vee_{i=1}^{\infty}(\lambda_i)\right)=1$ in (X,T). So $1-clint\left(\vee_{i=1}^{\infty}(\lambda_i)\right)=0$. Hence $intcl\left(\wedge_{i=1}^{\infty}(1-\lambda_i)\right)=0$, where (λ_i)'s are fuzzy nowhere dense sets in (X,T). □

Proposition 3.7. *If $\lambda\leq\mu$, where μ is a fuzzy set and λ is a fuzzy dense set in a fuzzy strongly irresolvable space (X,T), then $(1-\mu)$ is a fuzzy nowhere dense set in (X,T).*

Proof. Let λ be a fuzzy dense set in (X,T) such that $\lambda\leq\mu$. Since (X,T) is a fuzzy strongly irresolvable space, for the fuzzy dense set λ, $clint(\lambda)=1$ in (X,T). Now $\lambda\leq\mu$ implies that $clint(\lambda)\leq clint(\mu)$. Then $1\leq clint(\mu)$. Thus $clint(\mu)=1$ in (X,T). Now $1-clint(\mu)=0$ implies that $intcl(1-\mu)=0$ in (X,T). So $(1-\mu)$ is a fuzzy nowhere dense set in (X,T). □

Proposition 3.8. *If $\lambda\leq\mu$, where μ is a fuzzy set and λ is a fuzzy dense set in a fuzzy strongly irresolvable space (X,T), then $1-\mu$ is not a fuzzy open set in (X,T).*

Proof. Let λ be a fuzzy dense set in (X,T) such that $\lambda\leq\mu$. Since (X,T) is a fuzzy strongly irresolvable space, by Proposition 3.7 $(1-\mu)$ is a fuzzy nowhere dense set in (X,T). Then $intcl(1-\mu)=0$. But $int(1-\mu)\leq intcl(1-\mu)$ implies that

$int(1-\mu) \leq 0$. That is, $int(1-\mu) = 0$. Thus $(1-\mu)$ is not a fuzzy open set in (X,T). □

Proposition 3.9. *If λ is a fuzzy σ-nowhere dense set in a fuzzy strongly irresolvable space (X,T), then λ is a fuzzy nowhere dense and fuzzy semi-closed set in (X,T).*

Proof. Let λ be a fuzzy σ-nowhere dense set in (X,T). Then λ is a fuzzy F_σ-set in (X,T) such that $int(\lambda) = 0$. Now $cl(1-\lambda) = 1 - int(\lambda) = 1 - 0 = 1$, that is, $cl(1-\lambda) = 1$ in (X,T). Since (X,T) is a fuzzy strongly irresolvable space, for the fuzzy dense set $(1-\lambda)$, $clint(1-\lambda) = 1$ in (X,T). Thus $1 - intcl(\lambda) = 1$. That is, $intcl(\lambda) = 0$. So λ is a fuzzy nowhere dense set in (X,T). Also $intcl(\lambda) = 0$ implies that $intcl(\lambda) \leq \lambda$. Hence $intcl(\lambda) \leq \lambda$ implies that λ is a fuzzy semi-closed set in (X,T). Therefore λ is a fuzzy nowhere dense and fuzzy semi-closed set in (X,T). □

Proposition 3.10. *If λ is a fuzzy σ-first category set in a fuzzy strongly irresolvable space (X,T), then λ is a fuzzy first category set in (X,T).*

Proof. Let λ be a fuzzy σ-first category set in (X,T). Then $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy σ-nowhere dense sets in (X,T). Since (X,T) is a fuzzy strongly irresolvable space, by Proposition 3.9, (λ_i)'s are fuzzy nowhere dense sets in (X,T). Thus $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy nowhere dense sets in (X,T) implies that λ is a fuzzy first category set in (X,T). □

Proposition 3.11. *If λ is a fuzzy dense set in a fuzzy strongly irresolvable space (X,T), then $(1-\lambda)$ is a fuzzy nowhere dense and fuzzy semi-closed set in (X,T).*

Proof. Let λ be a fuzzy dense set in (X,T). Since (X,T) is a fuzzy strongly irresolvable space, for the fuzzy dense set λ, $clint(\lambda) = 1$ in (X,T). Then $1 - clint(\lambda) = 0$. Thus $intcl(1-\lambda) = 0$ implies that $(1-\lambda)$ is a fuzzy nowhere dense set in (X,T). Now $intcl(1-\lambda) \leq (1-\lambda)$. So $(1-\lambda)$ is a fuzzy semi-closed set in (X,T). Hence $(1-\lambda)$ is a fuzzy nowhere dense and fuzzy semi-closed set in (X,T). □

Proposition 3.12. *If $\lambda = \wedge_{i=1}^{\infty}(\lambda_i)$ is a fuzzy dense set, where (λ_i)'s are fuzzy sets in a fuzzy strongly irresolvable space (X,T), then $(1-\lambda)$ is a fuzzy first category set in (X,T).*

Proof. Let $\lambda = \wedge_{i=1}^{\infty}(\lambda_i)$ be a fuzzy dense set in (X,T). Then $cl\left(\wedge_{i=1}^{\infty}(\lambda_i)\right) = 1$. But $cl\left(\wedge_{i=1}^{\infty}(\lambda_i)\right) \leq \wedge_{i=1}^{\infty}\left(cl(\lambda_i)\right)$ in (X,T). Thus $1 \leq \wedge_{i=1}^{\infty}\left(cl(\lambda_i)\right)$. That is, $\wedge_{i=1}^{\infty}\left(cl(\lambda_i)\right) = 1$. So $cl(\lambda_i) = 1$ in (X,T). Since (X,T) is a fuzzy strongly irresolvable space, by Proposition 3.11, $(1-\lambda_i)$'s are fuzzy nowhere dense sets in (X,T). Now $\lambda = \wedge_{i=1}^{\infty}(\lambda_i)$ implies that $1-\lambda = 1 - \wedge_{i=1}^{\infty}(\lambda_i) = \vee_{i=1}^{\infty}(1-\lambda_i)$. That is, $1-\lambda = \vee_{i=1}^{\infty}(1-\lambda_i)$, where $(1-\lambda_i)$'s are fuzzy nowhere dense sets in (X,T). Hence $(1-\lambda)$ is a fuzzy first category set in (X,T). □

Proposition 3.13. *If each fuzzy dense set λ is a fuzzy open set in a fuzzy topological space (X,T), then (X,T) is a fuzzy strongly irresolvable space.*

Proof. Let λ be a fuzzy dense set in (X,T) such that $int(\lambda) = \lambda$. Then by Theorem 2.23, $(1-\lambda)$ is a fuzzy nowhere dense set in (X,T). Thus $intcl(1-\lambda) = 0$. So $1 - clint(\lambda) = 0$ and $clint(\lambda) = 1$, in (X,T). Hence for the fuzzy dense set λ in

(X,T), $clint(\lambda) = 1$ in (X,T). Therefore (X,T) is a fuzzy strongly irresolvable space. □

4. Fuzzy strongly irresolvable spaces and other fuzzy topological spaces

Proposition 4.1. *If a fuzzy topological space (X,T) is a fuzzy resolvable space, then (X,T) is not a fuzzy strongly irresolvable space.*

Proof. Let (X,T) be a fuzzy resolvable space. Then there exists a fuzzy dense set λ in (X,T) such that $cl(1-\lambda) = 1$ in (X,T). Thus $1 - int(\lambda) = 1$. So $int(\lambda) = 0$ in (X,T). Now $clint(\lambda) = cl(0) = 0 \neq 1$ in (X,T). Hence for the dense set λ in (X,T), $clint(\lambda) \neq 1$ implies that (X,T) is not a fuzzy strongly irresolvable space. □

Proposition 4.2. *If the fuzzy σ-first category space (X,T) is a fuzzy strongly irresolvable space, then (X,T) is a fuzzy first category space.*

Proof. Let (X,T) be a fuzzy σ-first category space. Then $\vee_{i=1}^{\infty}(\lambda_i) = 1$, where (λ_i)'s are fuzzy σ-nowhere dense sets in (X,T). Since (X,T) is a fuzzy strongly irresolvable space and (λ_i)'s are fuzzy σ-nowhere dense sets in (X,T), by Proposition 3.9, (λ_i)'s are fuzzy nowhere dense sets in (X,T). Thus $\vee_{i=1}^{\infty}(\lambda_i) = 1$, where (λ_i)'s are fuzzy nowhere dense sets in (X,T). So (X,T) is a fuzzy first category space. □

Proposition 4.3. *If the fuzzy almost irresolvable space (X,T) is a fuzzy strongly irresolvable space (X,T), then (X,T) is a fuzzy second category space.*

Proof. Let (X,T) be a fuzzy almost irresolvable space. Then $\vee_{i=1}^{\infty}(\lambda_i) \neq 1$, where the fuzzy set (λ_i)'s in (X,T) are such that $int(\lambda_i) = 0$. Since (X,T) is a fuzzy strongly irresolvable space, by Theorem 2.26, $int(\lambda_i) = 0$ in (X,T) implies that $intcl(\lambda_i) = 0$ in (X,T). Thus (λ_i)'s are fuzzy nowhere dense sets in (X,T). So $\vee_{i=1}^{\infty}(\lambda_i) \neq 1$, where (λ_i)'s are fuzzy nowhere dense sets in (X,T). Hence (X,T) is a fuzzy second category space. □

Proposition 4.4. *If the fuzzy σ-first category space (X,T) is a fuzzy strongly irresolvable space, then (X,T) is a fuzzy first category space.*

Proof. Let (X,T) be a fuzzy σ-first category space. Then $\vee_{i=1}^{\infty}(\lambda_i) = 1$, where (λ_i)'s are fuzzy σ-nowhere dense sets in (X,T). Since (X,T) is a fuzzy strongly irresolvable space and (λ_i)'s are fuzzy σ-nowhere dense sets in (X,T), by Proposition 3.9, (λ_i)'s are fuzzy nowhere dense sets in (X,T). Thus $\vee_{i=1}^{\infty}(\lambda_i) = 1$, where (λ_i)'s are fuzzy nowhere dense sets in (X,T). So (X,T) is a fuzzy first category space. □

Proposition 4.5. *If a fuzzy topological space (X,T) is a fuzzy σ-resolvable space, then (X,T) is not a fuzzy strongly irresolvable space.*

Proof. Let (X,T) be a fuzzy σ-resolvable space. Then $\vee_{\infty}^{\infty}(\lambda_i) = 1$, where (λ_i)'s are fuzzy dense sets in (X,T) such that $\lambda_i \leq (1-\lambda_j)$, for $i \neq j$. Now $cl(\lambda_i) \leq cl(1-\lambda_j)$. Thus $1 \leq cl(1-\lambda_j)$ in (X,T). That is, $cl(1-\lambda_j) = 1$ in (X,T). Now $clint(\lambda_j) = clint\,(1-(1-\lambda_j)) = 1 - intcl(1-\lambda_j) = 1 - int(1) = 1 - 1 = 0 \neq 1$. So $cl(\lambda_j) = 1$ and $clint(\lambda_j) \neq 1$ in (X,T). Hence (X,T) is not a fuzzy strongly irresolvable space. □

Proposition 4.6. *If $cl\,(\wedge_{i=1}^{\infty}(\lambda_i)) = 1$, where (λ_i)'s are fuzzy dense sets, in a fuzzy strongly irresolvable space (X,T), then (X,T) is a fuzzy Baire space.*

Proof. Let (λ_i)'s be fuzzy dense sets in (X,T) such that $cl\,(\wedge_{i=1}^{\infty}(\lambda_i)) = 1$. Since (X,T) is a fuzzy strongly irresolvable space, $clint(\lambda_i) = 1$ in (X,T). Then $1 - clint(\lambda_i) = 0$ in (X,T). Thus $intcl(1-\lambda_i) = 0$ in (X,T). This implies that $(1-\lambda_i)$'s are fuzzy nowhere dense sets in (X,T). Now $cl\,(\wedge_{i=1}^{\infty}(\lambda_i)) = 1$ implies that $1 - cl\,(\wedge_{i=1}^{\infty}(\lambda_i)) = 0$. So $int\,(\vee_{i=1}^{\infty}(1-\lambda_i)) = 0$ in (X,T). Hence $int\,(\vee_{i=1}^{\infty}(1-\lambda_i)) = 0$, where $(1-\lambda_i)$'s are fuzzy nowhere dense sets in (X,T). Therefore (X,T) is a fuzzy Baire space. □

Proposition 4.7. *If (λ_i)'s ($i = 1$ to ∞) are fuzzy dense sets in a fuzzy strongly irresolvable and fuzzy Baire space (X,T), then $cl\,(\wedge_{i=1}^{\infty}(\lambda_i)) = 1$ in (X,T).*

Proof. Let (λ_i)'s be fuzzy dense sets in (X,T). Since (X,T) is a fuzzy strongly irresolvable space, for the fuzzy dense sets (λ_i)'s in (X,T), $clint(\lambda_i) = 1$ in (X,T). Then $1 - clint(\lambda_i) = 0$. Thus $intcl(1-\lambda_i) = 0$, in (X,T). This implies that $(1-\lambda_i)$'s are fuzzy nowhere dense sets in (X,T). Since (X,T) is a fuzzy Baire space, $int\,(\vee_{i=1}^{\infty}(1-\lambda_i)) = 0$, where $(1-\lambda_i)$'s are fuzzy nowhere dense sets in (X,T). So $int\,(1 - \wedge_{i=1}^{\infty}(\lambda_i)) = 0$. Hence $1 - cl\,(\wedge_{i=1}^{\infty}(\lambda_i)) = 0$. Therefore $cl\,(\wedge_{i=1}^{\infty}(\lambda_i)) = 1$ in (X,T). □

Proposition 4.8. *If λ is a fuzzy first category set in a fuzzy Baire, fuzzy nodec and fuzzy strongly irresolvable space (X,T), then λ is a fuzzy closed set in (X,T).*

Proof. Let λ be a fuzzy first category set in (X,T). Then $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy nowhere dense sets in (X,T). Since (X,T) is a fuzzy nodec space, the fuzzy nowhere dense sets (λ_i)'s in (X,T) are fuzzy closed sets. Thus $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy closed sets in (X,T), implies that λ is a fuzzy F_σ-set in (X,T). Also, since (X,T) is a fuzzy Baire space, by Theorem 2.24, $int(\lambda) = 0$ in (X,T). So λ is a fuzzy F_σ-set such that $int(\lambda) = 0$ in (X,T). Hence λ is a fuzzy σ-nowhere dense set in (X,T). Since (X,T) is a fuzzy strongly irresolvable space, by Proposition 3.9, λ is a fuzzy nowhere dense set in (X,T). Again, since (X,T) is a fuzzy nodec space, the fuzzy nowhere dense set λ is a fuzzy closed set in (X,T). Therefore λ is a fuzzy closed set in (X,T). □

Proposition 4.9. *If a fuzzy topological space (X,T) is a fuzzy strongly irresolvable and fuzzy σ-Baire space, then (X,T) is a fuzzy Baire space.*

Proof. Let (X,T) be a fuzzy σ-Baire space. Then $int\,(\vee_{i=1}^{\infty}(\lambda_i)) = 0$, where (λ_i)'s are fuzzy σ-nowhere dense sets in (X,T). Since (X,T) is a fuzzy strongly irresolvable space, by Proposition 3.9, (λ_i)'s are fuzzy nowhere dense sets in (X,T). Thus $int\,(\vee_{i=1}^{\infty}(\lambda_i)) = 0$, where (λ_i)'s are fuzzy nowhere dense sets in (X,T). So (X,T) is a fuzzy Baire space. □

Proposition 4.10. *If $cl\,(\wedge_{i=1}^{\infty}(\lambda_i)) = 1$, where (λ_i)'s are fuzzy dense and fuzzy G_δ-sets, in a fuzzy strongly irresolvable space (X,T), then (X,T) is a fuzzy σ-Baire space.*

Proof. Suppose that $cl\,(\wedge_{i=1}^{\infty}(\lambda_i)) = 1$, where (λ_i)'s are fuzzy dense and fuzzy G_δ-sets in (X,T). Since (X,T) is a fuzzy strongly irresolvable space, for the fuzzy dense sets (λ_i)'s in (X, T), $clint(\lambda_i) = 1$, in (X, T). Then $1 - clint(\lambda_i) = 0$ and hence $intcl(1 - \lambda_i) = 0$ in (X, T). This implies that $(1 - \lambda_i)$'s are fuzzy nowhere dense sets in (X, T). Now $cl\,(\wedge_{i=1}^{\infty}(\lambda_i)) = 1$ implies that $1 - cl\,(\wedge_{i=1}^{\infty}(\lambda_i)) = 0$. Thus $int\,(\vee_{i=1}^{\infty}(1 - \lambda_i)) = 0$, in (X, T). But $int(1 - \lambda_i) \leq intcl(1 - \lambda_i)$ in (X, T). This implies that $int(1 - \lambda_i) \leq 0$. That is, $int(1 - \lambda_i) = 0$. Also, since (λ_i)'s are fuzzy G_δ-sets, $(1 - \lambda_i)$'s are fuzzy F_σ-sets, in (X, T). So $(1 - \lambda_i)$'s are fuzzy F_σ-sets in (X, T) such that $int(1 - \lambda_i) = 0$. Hence $(1 - \lambda_i)$'s are fuzzy σ-nowhere dense sets in (X, T). Therefore $int\,(\vee_{i=1}^{\infty}(1 - \lambda_i)) = 0$, where $(1 - \lambda_i)$'s are fuzzy σ-nowhere dense sets in (X, T) implies that (X, T) is a fuzzy σ-Baire space. □

5. Conclusions

In this paper, it is established that in fuzzy strongly irresolvable spaces, the fuzzy σ-nowhere dense sets are fuzzy nowhere dense and fuzzy σ-first category sets are fuzzy first category sets. The condition under which fuzzy topological spaces become fuzzy strongly irresolvable spaces is obtained by means of the fuzzy denseness of fuzzy open sets. It is proved that fuzzy first category sets are fuzzy closed sets in a fuzzy Baire, fuzzy nodec and fuzzy strongly irresolvable spaces. It is established that fuzzy resolvable and fuzzy σ-resolvable spaces are not fuzzy strongly irresolvable spaces. The conditions under which fuzzy strongly irresolvable spaces become fuzzy Baire spaces, fuzzy σ-Baire spaces are also obtained.

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References

[1] K. K. Azad, On fuzzy semicontinuity, fuzzy almost continuity and fuzzy weakly continuity, J. Math. Anal. Appl. 82 (1981) 14–32.
[2] G. Balasubramanian, Maximal fuzzy topologies, Kybernetika 31 (5) (1995) 459–464.
[3] Richard Bolstein, Sets of points of discontinuity, Proc. Amer. Math. Soc. 38 (1) (1973) 193–197.
[4] C. L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl. 24 (1968) 182–190.
[5] J. G. Ceder, On maximally resolvable spaces, Fund. Math. 55 (1964) 87–93.
[6] A. G. El'Kin, Ultrafilters and undecomposable spaces, Moscow Univ. math. Bull. 24 (5) (1969) 51–56.
[7] E. Hewitt, A problem of set theoretic topology, Duke Math. J. 10 (1943) 309–333.
[8] M. Katetov, On topological spaces containing no disjoint dense subsets, Math. Sbornik. (N.S). 21 (63) (1947) 3–12.
[9] David Rose, Kari Sizemore and Ben Thurston, Strongly irresolvable spaces, Inter. J. Math. and Math. Sci. 2006 (2006) 1–12.
[10] A. S. Bin Shahna, On fuzzy strong semicontinuity and fuzzy precontinuity, Fuzzy Sets and Systems 44 (1991) 303–308.
[11] G. Thangaraj and S. Anjalmose, On fuzzy Baire spaces, J. Fuzzy Math. 21 (3) (2013) 667–676.
[12] G. Thangaraj and S. Anjalmose, Some remarks on fuzzy Baire spaces, Scientia Magna 9 (1) (2013) 1–6.
[13] G. Thangaraj and G. Balasubramanian, On somewhat fuzzy continuous functions, J. Fuzzy Math. 11 (2) (2003) 725–736.
[14] G. Thangaraj and G. Balasubramanian, On fuzzy resolvable and fuzzy irresolvable spaces, Fuzzy Sets, Rough Sets and Multi-valued Operations and Applications 1 (2) (2009) 173–180.
[15] G. Thangaraj and E. Poongothai, On fuzzy σ-Baire spaces, Int. J. Fuzzy Math. Sys. 3 (4) (2013) 275-283.

[16] G. Thangaraj and V. Seenivasan, On fuzzy strongly irresolvable spaces, Proc. Nat. Conference on Fuzzy Math. Graph Theory (Jamal Mohamed College, Trichy, Tamilnadu, India), (2008).
[17] G. Thangaraj and D.Vijayan, On fuzzy almost resolvable and fuzzy almost irresolvable spaces, Int. J. Statistika and Mathematika 9 (1) (2014) 61–65.
[18] G. Thangaraj and D. Vijayan, On fuzzy σ-resolvable spaces, Int. J. Fuzzy Math. Sys. 4 (2) (2014) 209–219.
[19] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.

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